Kitaev model for qudits

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1 Qudits

Qubits are described mathematically by the vector space $\mathbb{C}^2$ with two matrices acting on it:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy $X^2 = Z^2 = I$ and $XZ = -ZX$. Note that (I, X, Z, XZ) is a basis of $\mathrm{End}(\mathbb{C}^2)$.

Now, qudits generalise this notion. Let d be a positive integer (we recover qubits when $d = 2$). Set $\xi = \exp(2\pi i/d)$. Consider the vector space $\mathbb{C}^d$ with the following two matrices acting on it:

$$X = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \xi & 0 & \cdots & 0 \\ 0 & 0 & \xi^2 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \xi^{d-1} \end{pmatrix}.$$

It is convenient to think about this in the following way. The vector space $\mathbb{C}^d$ has a basis $\{v_i, i \in \mathbb{Z}/d\mathbb{Z}\}$ and we have $X(v_i) = v_{i+1}$ and $Z(v_i) = \xi^i v_i$. Then it is easy to see that they satisfy $X^d = Z^d = I$ and $ZX = \xi XZ$. Moreover, the set $\{X^a Z^b \mid a, b \in \mathbb{Z}/d\mathbb{Z}\}$ is a basis of $\mathrm{End}(\mathbb{C}^d)$.

2 Construction of the model

Let $\Gamma = (S, E)$ be a finite oriented graph, S is the set of vertices, E is the set of arrows. Let $\Sigma = \Sigma_g$ be a connected orientable compact surface of genus $g \in \mathbb{Z}_{\geqslant 0}$. Assume that the graph Γ is inscribed in Σ (i.e., the surface Σ is a CW-complex whose 1-skeleton is Γ). Denote by F the set of faces (i.e., the connected components of $\Sigma \backslash \Gamma$). Note that the graph Γ is connected by construction.

We place a qudit on each arrow of the graph. The Hilbert space describing this system is $V = (\mathbb{C}^d)^{\otimes \# E}$. For each vertex $s \in S$ and each face $f \in F$ we define a vertex operator $A_s \colon V \to V$ and a face operator $B_f \colon V \to V$.

The vertex operator A_s acts by X on each qudit sitting on an arrow entering the vertex s, and by X^{-1} on each qudit sitting on an arrow leaving s. The face operator B_f acts by

Z on each qudit sitting on an arrow for which the face f lies on the right of the oriented arrow, and by Z^{-1} on each qudit sitting on an arrow for which the face f lies on the left of the oriented arrow.

$$X^{-1} \xrightarrow[Z]{Z^{-1}} X$$

Remark 2.1. Note that there is an involution $\psi\colon \mathbb{C}^d \to \mathbb{C}^d$, $v_i \mapsto v_{-i}$. It satisfies $X\psi = \psi X^{-1}$ and $Z\psi = \psi Z^{-1}$. This implies that the model does not change when we reverse an arrow in the graph and simultaneously apply ψ to the corresponding qudit.

3 Protected space

Set $VP = \bigcap_{s\in S} \mathrm{Ker}(A_s - I) \cap \bigcap_{f\in F} \mathrm{Ker}(B_f - I) \subset V$. Let us compute its dimension.

Proposition 3.1. *We have* $\dim VP = d^{2g}$.

Proof. Consider the cellular cochain complex

$$\mathcal{C}^0 \xrightarrow{d^0} \mathcal{C}^1 \xrightarrow{d^1} \mathcal{C}^2$$

for Σ over $\mathbb{Z}/d\mathbb{Z}$. We have

- $\mathcal{C}^0 = (\mathbb{Z}/d\mathbb{Z})$-linear combinations of elements of S,
- $\mathcal{C}^1 = (\mathbb{Z}/d\mathbb{Z})$-linear combinations of elements of E,
- $\mathcal{C}^2 = (\mathbb{Z}/d\mathbb{Z})$-linear combinations of elements of F.

The most obvious basis of V is given by $\mathcal{C}^1$. The basis of $\bigcap_{f\in F} \mathrm{Ker}(B_f - I) \subset V$, which is a subset of this basis, is given by $Z^1 \subset \mathcal{C}^1$, where $Z^1 = \mathrm{Ker}(d^1)$. Finally, the conditions coming from the A_s say that some basis elements must have the same coefficients. Then VP has a basis given by $Z^1/B^1 = H^1(\Sigma, \mathbb{Z}/d\mathbb{Z}) \simeq (\mathbb{Z}/d\mathbb{Z})^{2g}$, where $B^1 = \mathrm{Im}(d^0)$. □

4 Description of the eigenspaces

The $(\#S + \#F)$ operators A_s for $s \in S$ and B_f for $f \in F$ commute. They also satisfy

$$\prod_{s\in S} A_s = I, \qquad \prod_{f\in F} B_f = I, \qquad A_s^d = B_f^d = I.$$

Hence they are all diagonalizable in a common basis and their eigenvalues are dth roots of unity. We have a decomposition into a direct sum of common eigenspaces: $V = \bigoplus_\chi V_\chi$. The protected space VP is one of the V_χ.

Let us give a combinatorial description of each χ. Let us imagine that each vertex $s \in S$ may contain a number of copies of elementary electric charges e (with the convention that having d charges e is the same as having no charge at all, so the charges are considered modulo d). Similarly, we imagine that each face may contain a number of copies of magnetic charges m (also modulo d).

Now we interpret each χ as a distribution of charges. Fix χ, and take $v \in V_\chi$, $v \neq 0$. If we have $A_s v = \xi^p v$ and $B_f v = \xi^q v$, then we say that there are p elementary electric charges e at s (or simply the charge $p \cdot e$ at s), and that there are q elementary magnetic charges m at f (or simply the charge $q \cdot m$ at f). Additionally, the configuration of charges must be such that the total electric and magnetic charge is zero modulo d.

5 Path operators

For each (oriented) integer path (a path $t\colon s_1 \to s_2$ starting at a vertex s_1 and ending at a vertex s_2) we construct a path operator $S^Z(t)\colon V \to V$ in the following way: we apply Z to all qudits along the path (or Z^{-1} if the arrow goes in the opposite direction).

Similarly, for each (oriented) half-integer path (a path $t\colon f_1 \to f_2$ starting at a face f_1 and ending at a face f_2) we construct a path operator $S^X(t)\colon V \to V$ in the following way: we apply X to all qudits on edges whose directions point to the right of the path (and X^{-1} to those pointing to the left).

Let us see how these operators modify eigenspaces. Note that $S^Z(t)$ commutes with all A_s and B_f except for A_{s_1} and A_{s_2}. This means that when we have $S^Z(t) \cdot V_\chi \subset V_{\chi'}$, the configurations χ and χ' have the same charges except at s_1 and s_2. Moreover, we see that the path operator moves a charge $-e$ from s_1 to s_2.

Similarly, we see that for a half-integer path $t\colon f_1 \to f_2$, the path operator $S^X(t)$ moves a magnetic charge $-m$ from f_1 to f_2.

Remark 5.1. Since path operators act transitively on the set of χ's, all vector spaces V_χ have the same dimension. This allows another computation of the dimension of VP. The total dimension of V is $d^{\#E}$ and the number of possible χ's is $d^{\#S+\#F-2}$. Then the dimension of each V_χ is $d^{\#E-\#S-\#F+2} = d^{2g}$.

If Σ is a sphere ($g = 0$), then each V_χ is one-dimensional. Physically, this means that the configuration of charges completely determines the physical state of the system. For higher genus ($g > 0$), the spaces V_χ are degenerate (more than one-dimensional). This means that a configuration of charges does not determine the state of the physical system and that there are additional interior degrees of freedom.

Remark 5.2. Take a nonzero vacuum vector $v \in VP$. Take two different integer paths $t, t'\colon s_1 \to s_2$ with the same starting and ending points. Then we have $S^Z(t) \cdot v = S^Z(t') \cdot v$ whenever the paths t and t' are homotopic (this is always true for $g = 0$). However, for $g > 0$, a closed loop which is not homotopic to the identity can send a vacuum state to another vacuum state.

Now take a nonzero $v \in V_\chi$. Then we have $S^Z(t) \cdot v = S^Z(t') \cdot v$ whenever the paths t and t' are homotopic in such a way that the homotopy does not go through magnetic charges. However, if there is a homotopy traversing magnetic charges, then $S^Z(t) \cdot v$ and $S^Z(t') \cdot v$ would differ by a power of ξ depending on the magnetic charge.

In particular, when an e-particle is moved clockwise around an m-particle, this multiplies the state of the system by the phase factor ξ.